第 1 章

numbers and formulas

§ 1.1

numbers

This is a difficult question to answer. However, we use numbers everyday and we live without problems concerning numbers (except during exams). But is does not mean people from history have used numbers in their lives with ease. We will go over the general makings and the basic principles illustrated on graphs in this lesson.

1.1.1 Natural numbers

In Japanese;

一, 二, 三, 四, $\cdots$

In English:

one, two, three, four, $\cdots$

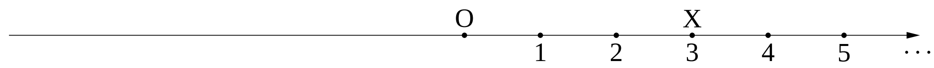
in such a way, counting numbers are taught in order. Putting objects in sequence to count them, in other words, is omitting other assets of each number. Three apples and three pencils are very different objects, but they use the exact same ’3’ for categorizing. There is no importance in number when it is about a sentimental pet such as a cat. Yet, there are values in number when there are no sentimental values such as shepherding 100 lambs. Without keeping count, one might be lost over night. *1 instead of numbering, you can name them. But it is painstaking when trying to name 100 lambs. Any number that can be used to count in order, greater than 1, is called natural number. For example:

1, 3, 42, 100, 23789

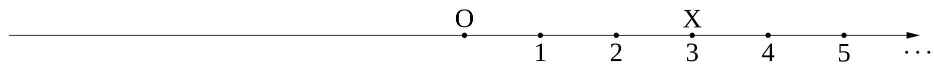
are all sufficient in defining natural numbers. Numbers such as $\frac{2}{3}$s and -3 are not considered natural numbers.

■Graphing Natural numbers

To illustrate natural numbers, consider the example below. First, put 0 on a straight line (arrow pointing right), and label each joint on each segment with equal length with the appropriate number.



D. Accordingly, each point represents a natural number. Line that present numbers is called a number line (shown above). When X is a point and the number is a, it is written as $X(a)$ as shown in the number line below.



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1.1.2 integers

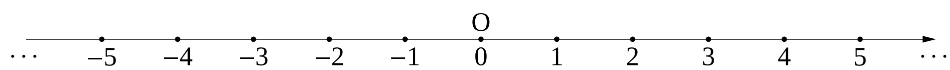
When considering natural numbers as counting numbers, there is no meaning in $3 - 5$ for example, because it is impossible to take 5 away from 3. However, if a person borrows 600yen from another person with 1000yen, then that person is left with 400yen. Moreover, if you borrow 1400yen from a person with 1000yen, you owe 400yen (formul?) which can be efficient. Such numbers; negative (−) natural numbers, 0, and natural numbers are altogether called integers. For example

$-2568, \quad -23, \quad -3, \quad 0, \quad 4, \quad 57$

are integers.

■Graphing integers

following the same steps as graphing natural numbers, label the points on the left side of 0 as well. Number line abbreviated.



1.1.3 rational number

Sorting 6 objects in groups of 3, $\frac{6}{3} = 2$ is formed. The 2 represents the 6 by 3 ratio. But 6 by 5 ratio cannot be described by integers in which results a new ratio: $\frac{6}{5}$.

Generally, $\frac{a}{b}$; where a is a integer and integer $b \neq 0$, is called rational number.

Rational numbers such as $\frac{4}{3}$ and $\frac{8}{6}$ both can be reduced to $\frac{4}{3}$ through reduction.Likewise, rational numbers that cannot be reduced is called irreducible fraction. Furthermore, integers can be expressed as $\frac{\text{integer}}{1}$, which implies integers can be rational numbers.

For example,

$$-\frac{8}{3},\quad -2,\quad 0,\quad \frac{11}{19},\quad \frac{18}{9},\quad 26$$

are rational numbers.

rational number

In the form $\frac{a}{b}$, where integer b is not integer a and 0, is called rational numbers.

0 is never used as the divisor. If $\frac{1}{0}$ is accepted, and if the ratio equals 0, then multiply both sides by 0

$$\frac{1}{\not{0}} \times \not{0} = 0 \times 0$$

$$\Leftrightarrow \quad 1 = 0 \qquad \leftarrow \text{reduced by } 0$$

Which creates a strange result.

■Complex Fraction

a ratio of rational number by another rational number, such as $\frac{\frac{2}{3}}{\frac{10}{7}}$ is written as⋯. Accordingly, when the divisor or the dividend is a rational number, it is called complex fraction. It can be reduced to a normal rational number by multiplying both the divisor and the dividend with the least common denominator. For example, if $\frac{\frac{2}{3}}{\frac{10}{7}}$, the least common denominator of 3 and 7 is 21. So multiply 21 on both the divisor and the dividend⋯.

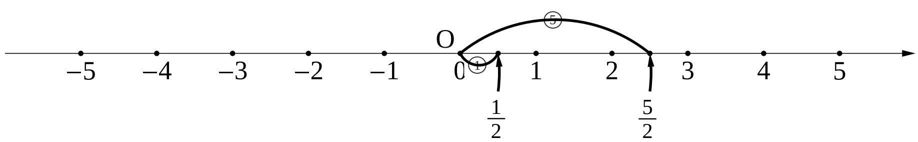
■Decimal number

Through computation on paper, rational numbers can be converted to decimal numbers as shown at the right. Some decimal numbers can be finite decimals such as $\frac{5}{4} = 1.25$, or infinite decimals, like $\frac{7}{22} = 0.3181818\cdots$. Circulating decimals are infinite decimals that repeat the same number. Use $\dot{}$ for showing which part of the decimal is repeated. For example, $\frac{7}{22}$

$$\frac{7}{22} = 0.3181818\cdots = 0.3\dot{1}\dot{8}$$

■Graphing Rational numbers

When putting $\frac{1}{2}$ on the number line below, for example, divide the segment between 0 and 1 equally and put $\frac{1}{2}$ on that joint. Also when putting $\frac{5}{2}$, think of it as $\frac{1}{2} \times 5$; so count up to five halves.



In order to graph a rational number such as ($\frac{a}{b}$), first divide the segment between 0 and 1 by the denominator (b) and count up a times with the distance of 0 to $\frac{1}{b}$. correspond to $\frac{1}{b} \times a$

■Density of Rational number

For example, between two rational numbers: $\frac{1}{3}$ and $\frac{2}{3}$,

mean-value of 10 and 20

$$\frac{1}{3} = \frac{10}{30} < \frac{15}{30} < \frac{20}{30} = \frac{2}{3}$$

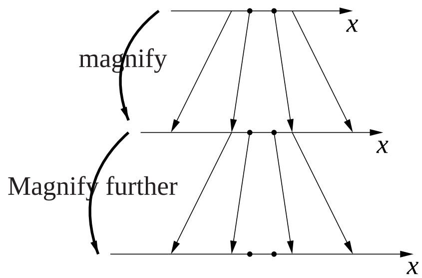
another rational number exist.

By using the formula,

mean-value of ad and bc

$$\frac{a}{b} = \frac{ad}{bd} < \frac{\frac{ad+bc}{2}}{bd} < \frac{bc}{bd} = \frac{c}{d}$$

There is always a rational number between two rational numbers



It is able to find a rational number between two other rational numbers.

Accordingly, the density of a rational number is that there are always another rational number existing between two others. So there are innumerable amount to rational numbers between another as shown if graphed:

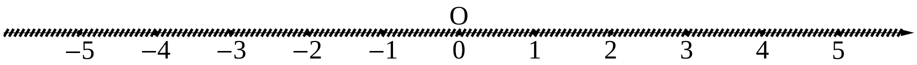


Image of a line stuffed with rational numbers

【Questions：Convert fractions to decimals and decimals to fractions】

Convert fractions to decimals and decimal numbers to fractions for the following questions (1) through (4)

(1) $\dfrac{9}{16}$ (2) $\dfrac{5}{37}$ (3) 0.625 (4) $0.\dot{4}2\dot{9}$

【Answer】

(1) (Use division) **0.5625**

(2) (Use division) $\mathbf{0.\dot{1}3\dot{5}}$

(3) 0.625 is $\dfrac{625}{1000}$ in fraction, then reduces to $\dfrac{5}{8}$.

(4) First, put

$$x = 0.429429429\cdots \quad \cdots\cdots\cdots\cdots ①$$

then multiply both sides by 1000 to get

$$1000x = 429.429429\cdots \quad \cdots\cdots\cdots\cdots ②$$

② − ①thus

$$\begin{array}{rrl} & 1000x &= 429.429429\cdots \\ -) & x &= 0.429429\cdots \\ \hline & 999x &= 429 \end{array}$$

Accordingly,, $x = \dfrac{429}{999} = \mathbf{\dfrac{143}{333}}$.

Computing circulating decimals just like x in algebraic equations maybe concerning, and this will be covered in FTEXT III.

1.1.4 Real Numbers

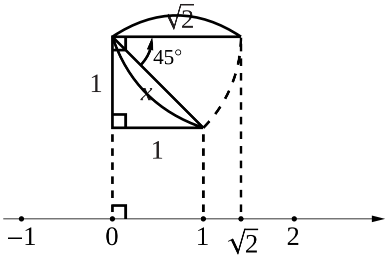
■Irrational Numbers

As mentioned before, there are full of rational numbers on a number line. Now let's see how if any point on a number line can be converted to a rational number.

For example, two adjacent sides of an isosceles triangle each with length 1 has a hypotenuse of x. According to the Pythagorean Theorem;

$$1^2 + 1^2 = x^2 \Leftrightarrow x^2 = 2$$

Consequently, $x = \sqrt{2}$ is the length and can be shown like this graph on the right.



【MEMORIZE : $\sqrt{2}$ is not a rational number】

Explain why $\sqrt{2}$ is not a rational number. Justify your answer.

When a conclusion is considered false, and while bringing justification that the conclusion is false, but come to wrong justification, then the first conclusion is justified. *** This theory is called reduction to absurdity. This theory is covered in FTEXT A.

【Answer】

Consider $\sqrt{2}$ is a rational number, so: $\sqrt{2} = \dfrac{a}{b}$ when a is an integer, and b is integer but isn't 0. When squaring both sides;

$$2 = \frac{a^2}{b^2}$$
$$\therefore\ 2b^2 = a^2 \quad \cdots\cdots\cdots\cdots ③$$

In this case, the left of the equal sign is a multiple two and therefore the a is also a multiple of two. Consequently, substitute a for $2a'$. Then,

$$2b^2 = (2a')^2$$
$$\Leftrightarrow\ 2b^2 = 4a'^2$$
$$\therefore\ b^2 = 2a'^2$$

The right side of the equation is a multiple of two so b squared should be a multiple of two, so b itself is a multiple of two. Then a and b are both multiples of two and it goes back to the first conclusion.

As a result, $\sqrt{2}$ is not a rational number ■

Therefore, there are numbers that exist on the number line that is not a rational number but is an irrational number.

■Irrational numbers that are not circulating decimals

The decimal number of $\sqrt{2}$ is between 1 and 2 since:$1^2 = 1,\ 2^2 = 4$ and $\sqrt{2}$ equals 2 when it is squared.

$1.4^2 = 1.96 < 2,\ \ 1.5^2 = 2.25 > 2$therefore,

$$1.4 < \sqrt{2} < 1.5$$

And since,$1.41^2 = 1.9881 < 2,\ \ 1.42^2 = 2.0164 > 2$

$$1.41 < \sqrt{2} < 1.42$$

can be said.

$$1.41421 < \sqrt{2} < 1.41422$$

Likewise, it is able to calculate $\sqrt{2}$ to the closest possible rational numbers endlessly. Irrational numbers,

Like $\sqrt{2}$ cannot have a circulating decimal because circulating decimals can be written in rational numbers.

■Real numbers

Real numbers are both rational and irrational numbers combined. Rational and irrational numbers fill the entire number line when graphed: *1

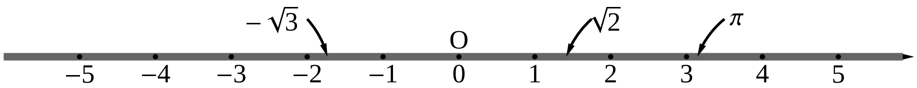


Image of a line graph packed with real numbers

For example:

$$-\sqrt{23},\quad -2\sqrt{3},\ \ 5\sqrt{2},\quad \sqrt{987}$$

Are irrational numbers. Also

$$\textbf{PI}\ \ \pi = 3.1415926\cdots$$
$$\textbf{Napier's constant}\ \ e = 2.7182818\cdots$$

Are considered irrational numbers.

When $a,\ b,\ x,$ are considered real numbers when the variables are not given a specific trait. Below shows a wrap up of the types of numbers.

*1 This is called **continuity** of real numbers. It will be discussed later in FTEXT III

Organization of numbers

- **Real numbers**
  - **Rational numbers**
    - **Integer**
      - *Naturalnumbers*
      - 0
      - *Negativeintegers*
    - *non − integerrationalnumbers*
      - *finitedecimal*
      - *Recurringdecimal* } infinite decimal
  - **Irrational numbers** ·············· *non − recurringinfinitedecimal* } infinite decimal

【Questions：numbers】

arrange the numbers into the following four categories

$3,\ -2,\ 0,\ \dfrac{2}{5},\ -\dfrac{2}{5},\ \sqrt{3},\ 1.\dot{5}\dot{2},\ \dfrac{36}{6},\ -\sqrt{16},\ \left(\sqrt{5}\right)^2,\ \pi^2$

(1) Find all natural numbers (2) Find all integers

(3) Find all rational numbers (4) Find all irrational numbers

【Answer】

(1) $\mathbf{3,\ \dfrac{36}{6},\ \left(\sqrt{5}\right)^2}$

(2) $\mathbf{3,\ -2,\ 0,\ \dfrac{36}{6},\ -\sqrt{16},\ \left(\sqrt{5}\right)^2}$

(3) $\mathbf{3,\ -2,\ 0,\ \dfrac{2}{5},\ -\dfrac{2}{5},\ 1.\dot{5}\dot{2},\ \dfrac{36}{6},\ -\sqrt{16},\ \left(\sqrt{5}\right)^2}$

(4) $\mathbf{\sqrt{3},\ \pi^2}$

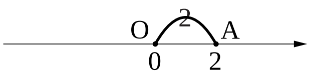
◂ $\dfrac{36}{6} = 6,\ \left(\sqrt{5}\right)^2 = 5$

◂ $-\sqrt{16} = -4$

◂ $1.\dot{5}\dot{2} = \dfrac{151}{99}$

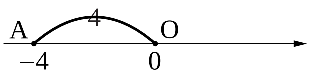
■absolute values

Absolute values are written with 1a1 such when finding the distance between point O and point $A(a)$.



For example:

$|2| = 2,\ \ |-4| = 4$



It can be concluded that absolute values are always positive.

values of positive numbers stays the same in absolute values

value of negative numbers are conversed in switching the sign in absolute values

Multiply by −1 when changing the positive/negative sign.

Absolute Values

Make two separate situations

$$|a| = \begin{cases} a & (\text{When} a \geqq 0) \\ -a & (\text{When} a < 0) \end{cases}$$

can be made. Consequently,

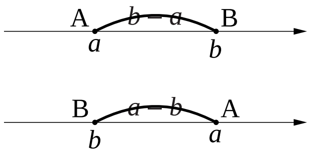
$$|a| \geqq 0 \ , \ |a| = |-a|$$

can be concluded.

To find the distance AB between point A(a) and point B(b),

AB = $|b - a|$

When either $a < b$ or $b < a$.



To find the distance between two points, subtract one from the other and then find the absolute value of the difference.

【MEMORIZE：Properties of absolute values】

ustify these conclusions about a and b. ($b \neq 0$ in question 3)

(1) $|a|^2 = a^2$ (2) $|ab| = |a||b|$ (3) $\left|\dfrac{a}{b}\right| = \dfrac{|a|}{|b|}$

Divide the work with the negative or the positive values inside the absolute value.

【Answer】

(1) i) When $a \geqq 0$, $|a| = a$

(Left-hand side) $= |a|^2 = a^2 =$ (Right-hand side)

ii) When $a < 0$, $|a| = -a$

(Left-hand side) $= |a|^2 = (-a)^2 = a^2 =$ (Right-hand side)

As of i) and ii), $|a|^2 = a^2$is concluded. ■

(2) First, consider $a > 0$

i) When $b \geqq 0$

$ab \geqq 0$, $|a| = a$, $|b| = b$ thus

(Left-hand side) $=|ab| = ab$

(Right-hand side) $=|a||b| = ab$

can be conluded.

ii) When $b = 0$

$ab = 0$, $|a| = a$, $|b| = 0$ thus

$$\text{(Left-hand side)} = |ab| = 0$$
$$\text{(Right-hand side)} = |a||b| = a \cdot 0 = 0$$

can be conluded.

iii) When $b < 0$

Since $ab < 0$, $|a| = a$, $|b| = -b$ thus

$$\text{(Left-hand side)} = |ab| = -ab$$
$$\text{(Right-hand side)} = |a||b| = a(-b) = -ab$$

can be conluded.

In case of $a < 0$ の場合も同様に示すことができる.
また, $a = 0$ の場合には両辺 0 になり成立する.
以上より, $|ab| = |a||b|$ が成り立つ. ■

(3) First,

$$\left|\frac{1}{b}\right| = \frac{1}{|b|} \quad \cdots\cdots\cdots\cdots\cdots\cdots\cdots ①$$

can be written.

i) When $b > 0$

$\dfrac{1}{b} > 0$, $|b| = b$ so

$$\text{(left side of ①)} = \left|\frac{1}{b}\right| = \frac{1}{b}$$
$$\text{(right side of ①)} = \frac{1}{|b|} = \frac{1}{b}$$

is concluded.

ii) $b < 0$ のとき

$\dfrac{1}{b} < 0$, $|b| = -b$ so

$$\text{(left side of ①)} = \left|\frac{1}{b}\right| = -\frac{1}{b}$$
$$\text{(right side of ①)} = \frac{1}{|b|} = \frac{1}{-b} = -\frac{1}{b}$$

is calculated.

As of i) and ii), ① is concluded. Consequently,

$$\left|\frac{a}{b}\right| = \left|a \cdot \frac{1}{b}\right|$$
$$= |a|\left|\frac{1}{b}\right|$$

◂ (2)is used

$$= |a| \frac{1}{|b|}$$ ◂ ①is used

$$= \frac{|a|}{|b|}$$

so, $\left|\frac{a}{b}\right| = \frac{|a|}{|b|}$ is concluded. ■

Properties of Absolute values

Justify these conclusions about a and b. ($b \neq 0$ in iii))

i) $|a|^2 = a^2$ ii) $|ab| = |a||b|$ iii) $\left|\frac{a}{b}\right| = \frac{|a|}{|b|}$

Absolute value equations are used in alternating equations. (1) is for squaring, with or without absolute value, (2) is for multiplying inside one or separate absolute values, (3) is for division in one or separate absolute values.

§ 1.2

Calculating Expressions

Expressions that contains variables can be divided into two kinds.

1.2.1 Integral Expression

■Monomial

A monomial is a group of numbers and variables that are multiplied like $3abx^2$. The numbers in the expression are called **coefficients** and the sum of all variables that are being multiplied is **degree**. For example, $3abx^2$ has a coefficient of 3 and a degree of 4 (see right).

coefficient

$$3 \quad abx^2$$

four variables are multiplied so the degree is 4

Especially, numbers like 1 or -3 are considered as unary expressions containing no variables, and the degree is 0 (all numbers except 0). Also, the unary expression of 0 is considered to have no degree at all.

■Polynomial

Just like $2a - 3b^2 + ab$, a group of monomials added or subtracted is called a **polynomial**. Each monomial in the polynomial is a **term**. Polynomial $(-a^2) + (4ab) + (-6b^2)$ is simply written $-a^2 + 4ab - 6b^2$.

For example, polynomial$5a^2b + 3ab - a^2b + 2ab$

similar terms
$$5a^2b + 3ab - a^2b + 2ab$$
similar terms

$=(5-1)a^2b + (3+2)ab$ ← Similar terms are assembled by computing the coefficients

$=4a^2b + 5ab$

simplified and gathered by similar terms

In a polynomial, the number of variables multiplied in the term with the most multiplied variable is the **degree** of the polynomial. For example, at the right, $4a^2b + 5ab$ has a degree of 3.

$$4a^2b + 5ab$$
degree of 3 degree of 2

degree of the polynomial is 3 (the larger degree of the t

■Integral expression

Both monomial and polynomial are **integral expressions** and the degree of the expression with degree n is called the **expression degree of *n***. For example, $4x-3$ has a degree of 1 and $3x^2 - 4x + 5$ has a degree of 2.

In a polynomial, the terms can be arranged by focusing on a certain variable. The expression is described according to the certain variable and the term without the variable is called the **constant term**. For example, $3x^3y + bx + c$, when the expression is based on x; the expression has a degree of 3, ay is the coefficient of x^3, b is the coefficient of x and c is the constant term.

Moreover, if the expression is based on y, then it has a degree of 1, $ax3$ is the coefficient of y and $bx + c$ is the constant term.（Refer to a right picture）

When it is an expression of x

coefficient

ay x^3 +
Degree of 3

coefficient constant term

b x + c
Degree of 1 Degree of 0

When considered as an expression of y

coefficient constant term

ax^3 y + $bx + c$
Degree of 1 Degree of 0

■Descending / Ascending Order of power

Magnitude of the degree of an expression is described as **high** or **low**. For example, $-7 + 4x3 - 3x2 + x$

when arranged from highest degree: $4x^3 - 3x^2 + x - 7$

when arranged from lowest degree: $-7 + x - 3x^2 + 4x^3$

Consequently, when an expression is arranged from highest degree,
It is arranged by the **descending order of power**.
And when it is arranged from lowest degree,
it is arranged by the **ascending order of power**.

【Problems: Degree of Integral expression #1 **】**

For the following integral expressions, find what degree it is when based on the variable inside [ ] and its coefficient.

(1) $3x^4y^5$ [x], [y], [$x$and$y$] (2) $2abxy^2$ [x], [y], [$x$and$y$]

【Answers】

(1) i) when based on x, has a degree of **4**, coefficient of $\mathbf{3y^5}$

ii) when based on y, has a degree of **5**, coefficient of $\mathbf{3x^4}$

iii) when based on x and y, degree of **9**, coefficient of **3**

(2) i) when based on x, degree of **1**, coefficient of $\mathbf{2aby^2}$

ii) when based on y, degree of **2**, coefficient of $\mathbf{2abx}$

iii) when based on x and y, degree of **3**, coefficient of $\mathbf{2ab}$

【Problem: Degree of Integral expression #2】

Simplify the following expressions below and find the degree of the expression when based on the variable [ ]

(1) $x^3 + xy^2 - 3xy^2$ [x], [y] (2) $x^4 + (a^2 - a)x^3y + 2bxy^2 + ax^3y - 4bxy^2$ [x], [y]

【Answers】

(1) When simplified,

$$x^3 + xy^2 - 3xy^2 = \mathbf{x^3 - 2xy^2}$$

i) when based on x, the term x^3 has degree of 3 and term $-2xy^2$ has a degree of 1 so it has a third expression (expression of degree 3).

◀ Choose the degree of the term with the largest degree

ii) When based on y, the term x^3 has degree of 0 (constant term), term $-2xy^2$ has a degree of 2, so it has an expression of degree 2.

◀ Choose the degree of the term with the largest degree

(2) When simplified,

$$\begin{aligned} & x^4 + (a^2 - a)x^3y + 2bxy^2 + ax^3y - 4bxy^2 \\ =& x^4 + (a^2 - a + a)x^3y + (2 - 4)bxy^2 \\ =& \mathbf{x^4 + a^2x^3y - 2bxy^2} \end{aligned}$$

i) when based on x, the term x^4 has degree of 4, 3 for a^2x^3y, 1 for $2bxy^2$, so it has an expression of degree 4.

◂ Choose the degree of the term with the largest degree

ii) When based on y, term x^4 has degree of 0, 1 for a^2x^3y and 2 for $2bxy^2$, so it has an expression of degree 2.

◂ Choose the degree of the term with the largest degree

【Problem: Descending order of power】

Arrange the following integral expressions in descending order of power and name each coefficient for each term.

(1) $3x^2 - 12xy + 4 + 3x^2 - 2x + 5$

(2) $2x^2 + 2y^2 - 3xy + 4y^2 + 2xy - x^2$

【Answer】

(1) First simplify similar terms, then, put in order.

$$3x^2 - 12xy + 4 + 3x^2 - 2x + 5$$
$$=(3+3)x^2 + (-12y - 2)x + (4+5)$$
$$=\mathbf{6x^2 - (12y + 2)x + 9}$$

Consequently, x^2 has a coefficient of **6**, coefficient $-(12y + 2)$ for x and **9** is the constant term.

(2) First simplify similar terms, then, put in order.

$$2x^2 + 2y^2 - 3xy + 4y^2 + 2xy - x^2$$
$$=(2-1)x^2 + (2-3)xy + (2+4)y^2$$
$$=x^2 - yx + 6y^2$$

Consequently, x^2 has a coefficient of **1**, coefficient $-y$ for x, and $\mathbf{6y^2}$ is the constant term.

1.2.2 Adding & Subtracting Integral expressions

For example, when $A = 3x^2-2x+1$, $B = 2x^2+7x-3$, the sum and difference of expressions A and B can be solved accordingly:

i) **Adding Integral expressions**

$$A + B =(3x^2 - 2x + 1) + (2x^2 + 7x - 3)$$
$$=3x^2 - 2x + 1 + 2x^2 + 7x - 3 \qquad \leftarrow \text{rentheses taken out}$$
$$=(3+2)x^2 + (-2+7)x + (1-3) \qquad \leftarrow \text{Gathered similar terms}$$
$$=5x^2 + 5x - 2$$

ii) **Subtracting Integral expressions**

$$A - B =(3x^2 - 2x + 1) - (2x^2 + 7x - 3)$$
$$=3x^2 - 2x + 1 - 2x^2 - 7x + 3 \qquad \leftarrow \text{rentheses taken out}$$
$$=(3-2)x^2 + (-2-7)x + (1+3) \qquad \leftarrow \text{Gathered similar terms}$$
$$=x^2 - 9x + 4$$

The expressions can be computed vertically by lining up similar terms.

i)

$$\begin{array}{rrrrrrrr} & 3x^2 & - & 2x & + & 1 \\ +) & 2x^2 & + & 7x & - & 3 \\ \hline & 5x^2 & + & 5x & - & 2 \end{array}$$

ii)

$$\begin{array}{rrrrrrrr} & 3x^2 & - & 2x & + & 1 \\ -) & 2x^2 & + & 7x & - & 3 \\ \hline & x^2 & - & 9x & + & 4 \end{array}$$

1.2.3 Multiplying Integral Expressions

■Law of powering

For example, $2 \times 2 \times 2 = 2^3$, $6 \times 6 \times 6 \times 6 = 6^4$ when real number a is multiplied by itself n times, it is written:

$$\underbrace{a \times a \times \cdots \times a}_{n \text{ numbers of } a} = a^n$$

and is read "a to the nth power". n is called the **exponent**. When $a^1 = a$, a^2 is called **square** and a^3 is called **cube**. $a, a^2, a^3, \cdots$. are called **powers** of a.

From the following powers

i) $3^2 \times 3^4 = (\underbrace{3 \times 3}_{2 \text{ numbers of } 3}) \times (\underbrace{3 \times 3 \times 3 \times 3}_{4 \text{ numbers of } 3}) = 3^6 \; (= 3^{2+4})$

ii) $(3^2)^4 = (\underbrace{3 \times 3}_{2 \text{ numbers of } 3}) \times (\underbrace{3 \times 3}_{2 \text{ numbers of } 3}) \times (\underbrace{3 \times 3}_{2 \text{ numbers of } 3}) \times (\underbrace{3 \times 3}_{2 \text{ numbers of } 3}) = 3^8 \; (= 3^{2\times4})$

iii) $(2 \times 3)^4 = (2 \times 3) \times (2 \times 3) \times (2 \times 3) \times (2 \times 3) = 2^4 \times 3^4$

Thus, the power law can be concluded:

Power law

The following are properties when m, n are whole numbers.

i) $a^m a^n = a^{m+n}$ ii) $(a^m)^n = a^{mn}$ iii) $(ab)^n = a^n b^n$

m and n are whole numbers for this lesson but real numbers and decimal numbers will be used later in FTEXT II.

■multiplying integral expressions

The distributive properties $A(B + C) = AB + AC$, $(A + B)C = AC + BC$ can also be used when A, B, C are integral expressions. For example $(x^2 + 3)(x^2 - 4x + 5)$ can be solved using the distributive property.

$(x^2 + 3)(x^2 - 4x + 5)$

$= x^2(x^2 - 4x + 5) + 3(x^2 - 4x + 5)$ ← $x^2 - 4x + 15$ is distributed as one component

$= x^4 - 4x^3 + 5x^2 + 3x^2 - 12x + 15$ ← Got rid of parentheses

$= x^4 - 4x^3 + 8x^2 - 12x + 15$ ← Compute similar terms and put them in order of powers

Expansion is calculating the product of the expressions and arranging monomials into addition, as performed just above.

1.2.4 Formulas for multiplying integral expressions

Try to memorize all the algebraic formulas and its expansions that will be introduced from now on. Just like memorizing multiplication up to 9x9, if you keep on practicing and try to memorize these algebraic formulas, you will be able to compute expressions with ease.

■Squared formulas

When expanding $(a + b)^2$

$$(a + b)^2 = (a + b)\ (a + b)$$

(①, ②, ③, ④)

$$= \overset{①}{a^2} + \overset{②}{ab} + \overset{③}{ba} + \overset{④}{b^2}$$

$$= a^2 + 2ab + b^2$$

	a	b
a	a^2	ab
b	ba	b^2

$(a + b)^2 = a^2 + 2ab + b^2$. In the same way, $(a - b)^2 = a^2 - 2ab + b^2$.. For example, $(3x + 2)^2$ can be solved using the formula.

i) Efficient way (○)

$$(3x + 2)^2 = \underbrace{9x^2 + 2 \cdot 3x \cdot 2 + 4}_{\text{Abbreviate when you get used to it}}$$

$$= 9x^2 + 12x + 4$$

ii) typical way (×)

$$(3x + 2)^2 = (3x + 2)(3x + 2)$$

$$= 9x^2 + 6x + 6x + 4$$

$$= 9x^2 + 12x + 4$$

Squared Formula

1° $(a + b)^2 = a^2 + 2ab + b^2$

2° $(a - b)^2 = a^2 - 2ab + b^2$

■Formulas: Product of addition and subtraction of integral expressions

When expanding $(a + b)(a - b)$

$$(a + b)(a - b) = (a + b)\ (a - b)$$

(①, ②, ③, ④)

$$= \overset{①}{a^2} - \overset{②}{ab} + \overset{③}{ba} - \overset{④}{b^2}$$

	a	$-b$
a	a^2	$-ab$
b	ba	b^2

$$= a^2 - b^2$$

Accordingly,$(a+b)(a-b) = a^2 - b^2$ is formed

Using this, $(5x+2y)(5x-2y)$ is solved below.

i) Efficient way（○）

$$(5x+2y)(5x-2y)$$
$$= 25x^2 - 4y^2$$

ii) typical way（×）

$$(5x+2y)(5x-2y)$$
$$= 25x^2 - 10xy + 10yx - 4y^2$$
$$= 25x^2 - 4y^2$$

Formulas: Product of addition and subtraction of integral expressions

3° $(a+b)(a-b) = a^2 - b^2$

■Formula: Product of linear expression

When expanding $(ax+b)(cx+d)$

$$(ax+b)(cx+d) = (ax+b)\ (cx+d)$$

$$\overset{①}{=\ acx^2} + \overset{②}{adx} + \overset{③}{bcx} + \overset{④}{bd}$$
$$= acx^2 + (ad+bc)x + bd$$

	cx	d
ax	acx^2	adx
b	bcx	bd

$(ax+b)(cx+d) = acx^2+(ad+bc)x+bd$. When $a = c = 1$, then $(x+b)(x+d) = x^2+(b+d)x+bd$. Using this, $(2x+3y)(5x-4y)$ can be solved accordingly.

i) Efficient way（○）

$$(2x+3y)(5x-4y)$$
$$= \underbrace{10x^2 + (-8+15)xy - 12y^2}_{\text{慣れると省略できる}}$$
$$= 10x^2 + 7xy - 12y^2$$

ii) typical way（×）

$$(2x+3y)(5x-4y)$$
$$= 10x^2 - 8xy + 15yx - 12y^2$$
$$= 10x^2 + 7xy - 12y^2$$

Formula: Product of linear expression

4° $(x+b)(x+d) = x^2 + (b+d)x + bd$

5° $(ax+b)(cx+d) = acx^2 + (ad+bc)x + bd$

■Formula for squaring three terms

When expanding$(a+b+c)^2$

$$\begin{aligned}(a+b+c)^2 &= \{(a+b)+c\}^2\\ &= (a+b)^2+2(a+b)c+c^2\\ &= a^2+2ab+b^2+2ca+2bc+c^2\\ &= a^2+b^2+c^2+2ac+2bc+2ca\end{aligned}$$

	a	b	c
a	a^2	ab	ac
b	ba	b^2	bc
c	ca	cb	c^2

$(a+b+c)^2 = a^2+b^2+c^2+2ab+2bc+2ca$. Using this, $(2x+y-3)^2$ can be solved accordingly.

i) Efficient way（○）

$$\begin{aligned}&(2x+y-3)^2\\ =\ & \underbrace{4x^2+y^2+9+2\cdot 2xy+2\cdot y(-3)+2\cdot(-3)2x}_{\text{Abbreviate when you get used to it}}\\ =\ & 4x^2+y^2+9+4xy-6y-12x\end{aligned}$$

ii) typical way（×）

$$\begin{aligned}&(2x+y-3)^2\\ &= (2x+y-3)(2x+y-3)\\ &= 4x^2+2xy-6x+2yx+y^2-3y-6x-3y+9\\ &= 4x^2+y^2+9+4xy-6y-12x\end{aligned}$$

Formula for squaring 3 terms

6° $(a+b+c)^2 = a^2+b^2+c^2+2ab+2bc+2ca$

■Cubic formula #1

When expanding $(a+b)^3$

$$\begin{aligned}(a+b)^3 &= (a+b)(a+b)^2\\ &= (a+b)(a^2+2ab+b^2)\\ &= (a+b)\ (a^2+2ab+b^2)\\ &= \overset{①}{a^3}+\overset{②}{2a^2b}+\overset{③}{ab^2}+\overset{④}{ba^2}+\overset{⑤}{2ab^2}+\overset{⑥}{b^3}\\ &= a^3+3a^2b+3ab^2+b^3\end{aligned}$$

	a^2	$2ab$	b^2
a	a^3	$2a^2b$	ab^2
b	ba^2	$2ab^2$	b^3

$(a+b)^3 = a^3+3a^2b+3ab^2+b^3$. In the same way $(a-b)^3 = a^3-3a^2b+3ab^2-b^3$. Using this, $(2x+y)^3$ can be solved accordingly.

i) Efficient way（○）

$$(2x+y)^3$$
$$= \underbrace{8x^3 + 3\cdot 4x^2y + 3\cdot 2xy^2 + y^3}_{\text{Abbreviate when you get used to it}}$$
$$= 8x^3 + 12x^2y + 6xy^2 + y^3$$

ii) typical way（×）

$$(2x+y)^3$$
$$= (2x+y)(2x+y)^2$$
$$= (2x+y)(4x^2+4xy+y^2)$$
$$= 8x^3 + 8x^2y + 2xy^2 + 4x^2y + 4xy^2 + y^3$$
$$= 8x^3 + 12x^2y + 6xy^2 + y^3$$

Cubic formula #1

7° $(a+b)^3 = a^3 + 3a^2b + 3ab^2 + b^3$

8° $(a-b)^3 = a^3 - 3a^2b + 3ab^2 - b^3$

For reference:

$$(a+b)^4 = a^4 + 4a^3b + 6a^2b^2 + 4ab^3 + b^4$$
$$(a+b)^5 = a^5 + 5a^4b + 10a^3b^2 + 10a^2b^3 + 5ab^4 + b^5$$

The ordinary expansion $(a+b)^n$ will be covered in FTEXT A.

■Cubic Formula #2

When expanding $(a+b)(a^2-ab+b^2)$

$$(a+b)(a^2-ab+b^2) = (a+b)\ (a^2-ab+b^2)$$
$$= \overset{①}{a^3} - \overset{②}{a^2b} + \overset{③}{ab^2} + \overset{④}{ba^2} - \overset{⑤}{ab^2} + \overset{⑥}{b^3}$$
$$= a^3 + b^3$$

	a^2	$-ab$	b^2
a	a^3	$-a^2b$	ab^2
b	ba^2	$-ab^2$	b^3

$(a+b)(a^2-ab+b^2) = a^3+b^3$. Also, $(a-b)(a^2+ab+b^2) = a^3-b^3$. Using this, $(3x+1)(9x^2-3x+1)$ is solved accordingly.

i) Efficient way（○）

$$(3x+1)(9x^2-3x+1)$$
$$= \underbrace{(3x+1)\left\{(3x)^2 - (3x)\cdot 1 + 1^2\right\}}_{\text{Abbreviate when you get used to it}}$$
$$= 27x^3 + 1$$

ii) typical way（×）

$$(3x+1)(9x^2-3x+1)$$
$$= 27x^3 - 9x^2 + 3x + 9x^2 - 3x + 1$$
$$= 27x^3 + 1$$

Cubic Formula #2

9° $(a+b)(a^2-ab+b^2)=a^3+b^3$

10° $(a-b)(a^2+ab+b^2)=a^3-b^3$

Typically, this $(a+b)(a^2-ab+b^2)$ and $(a-b)(a^2+ab+b^2)$ is not expanded further. On the other hand, formulas 9° and 10° are used in factorizing from right to left, which will be covered right after this lesson.

【Practice Problems: Expanding Integral expressions #1**】**

Expand the following expressions.

(1) $\left(\frac{1}{2}x+\frac{1}{3}y\right)^2$ (2) $\left(3a-\frac{1}{2}b\right)^2$

(3) $(2x-5y)(2x+5y)$ (4) $(-2ab+3c)(2ab+3c)$

(5) $(x+5)(x-8)$ (6) $(a^2-3)(a^2+7)$

(7) $(2x+1)(x-3)$ (8) $(3a-2)(4a+1)$

(9) $(3a-b+3c)^2$ (10) $(a^2+a-1)^2$

(11) $(2x+1)^3$ (12) $(3a-2)^3$

(13) $(x+2)(x^2-2x+4)$ (14) $(ab-3)(a^2b^2+3ab+9)$

【Answers】

(1) $\left(\frac{1}{2}x+\frac{1}{3}y\right)^2=\boldsymbol{\frac{1}{4}x^2+\frac{1}{3}xy+\frac{1}{9}y^2}$ ◂ Squared Formula(p.17)

(2) $\left(3a-\frac{1}{2}b\right)^2=\boldsymbol{9a^2-3ab+\frac{1}{4}b^2}$ ◂ Squared Formula(p.17)

(3) $(2x-5y)(2x+5y)=\boldsymbol{4x^2-25y^2}$ ◂ Formulas: Product of addition and subtraction of integral expressions(p.17)

(4) $(-2ab+3c)(2ab+3c)=\boldsymbol{9c^2-4a^2b^2}$ ◂ Formulas: Product of addition and subtraction of integral expressions(p.17)

(5) $(x+5)(x-8)=\boldsymbol{x^2-3x-40}$ ◂ Formula: Product of linear expression(p.18)

(6) $(a^2-3)(a^2+7)=\boldsymbol{a^4+4a^2-21}$ ◂ Formula: Product of linear expression(p.18)

(7) $(2x+1)(x-3)=\boldsymbol{2x^2-5x-3}$ ◂ Formula: Product of linear expression(p.18)

(8) $(3a-2)(4a+1)=\boldsymbol{12a^2-5a-2}$ ◂ Formula: Product of linear expression(p.18)

(9) $(3a-b+3c)^2=\boldsymbol{9a^2+b^2+9c^2-6ab-6bc+18ca}$ ◂ Formula for squaring three terms(p.19)

(10) $(a^2+a-1)^2=a^4+a^2+1+2a^3-2a-2a^2$

$=\boldsymbol{a^4+2a^3-a^2+1}$ ◂ Formula for squaring three terms(p.19)

(11) $(2x+1)^3=\boldsymbol{8x^3+12x^2+6x+1}$ ◂ Cubic formula #1(p.19)

(12) $(3a-2)^3=\boldsymbol{27a^3-54a^2+36a-8}$ ◂ Cubic formula #1(p.19)

(13) $(x+2)(x^2-2x+4)=\boldsymbol{x^3+8}$ ◂ Cubic Formula #2(p.20)

(14) $(ab-3)(a^2b^2+3ab+9)=\boldsymbol{a^3b^3-27}$ ◂ Cubic Formula #2(p.20)

【Practice problems: Expanding Integral expressions #2】

Expand the following expressions

(1) $(x+y+z)(x+y-z)$
(2) $(2a-b+c)(2a+b+c)$
(3) $(x+y+z+w)(x+y-z-w)$
(4) $(a^2+a-1)(a^2-a-1)$
(5) $(x+1)(x-1)(x^2+1)$
(6) $(a-2)(a+2)(a^2+4)(a^4+16)$
(7) $(x-1)(x-3)(x+3)(x+1)$
(8) $(a-1)(a-2)(a^2-3a)$
(9) $(x+y)(x-y)(x^2+xy+y^2)(x^2-xy+y^2)$
(10) $(a+b)^2(a-b)^2(a^4+a^2b^2+b^4)^2$

【Answers】

(1)

$(x+y+z)(x+y-z) = (x+y)^2 - z^2$

$= \mathbf{x^2 + 2xy + y^2 - z^2}$

◀ Formulas: Product of addition and subtraction of integral expressions(p.17)
◀ **Squared Formula**(p.17)

(2)

$(2a-b+c)(2a+b+c) = (2a+c)^2 - b^2$

$= \mathbf{4a^2 + 4ac + c^2 - b^2}$

◀ Formulas: Product of addition and subtraction of integral expressions(p.17)
◀ **Squared Formula**(p.17)

(3)

$(x+y+z+w)(x+y-z-w)$

$=(x+y)^2 - (z+w)^2$

$= \mathbf{x^2 + 2xy + y^2 - z^2 - 2zw - w^2}$

◀ **Squared Formula**(p.17)
◀ **Squared Formula**(p.17)

(4)

$(a^2+a-1)(a^2-a-1) = (a^2-1)^2 - a^2$

$= a^4 - 2a^2 + 1 - a^2 = \mathbf{a^4 - 3a^2 + 1}$

◀ Formulas: Product of addition and subtraction of integral expressions(p.17)
◀ **Squared Formula**(p.17)

(5)

$(x+1)(x-1)(x^2+1) = (x^2-1)(x^2+1)$

$= \mathbf{x^4 - 1}$

◀ **Squared Formula**(p.17)
◀ **Squared Formula**(p.17)

(6)

$(a-2)(a+2)(a^2+4)(a^4+16)$

$=(a^2-4)(a^4+4)(a^4+16)$

$=(a^4-16)(a^4+16) = \mathbf{a^8 - 256}$

◀ **Squared Formula**(p.17)
◀ **Squared Formula**(p.17)

(7)

$(x-1)(x-3)(x+3)(x+1) = (x^2-1)(x^2-9)$

$= \mathbf{x^4 - 10x^2 + 9}$

◀ **Squared Formula**(p.17)
◀ Formulas: Product of addition and subtraction of integral expressions(p.17)

(8)

$(a-1)(a-2)(a^2-3a) = (a^2-3a+2)(a^2-3a)$

$=(a^2-3a)^2 + 2(a^2-3a)$

◀ Formulas: Product of addition and subtraction of integral expressions(p.17)
◀ Expanded

$=a^4 - 6a^3 + 9a^2 + 2a^2 - 6a$

$=\boldsymbol{a^4 - 6a^3 + 11a^2 - 6a}$

(9)

$(x + y)(x - y)(x^2 + xy + y^2)(x^2 - xy + y^2)$

$=(x - y)(x^2 + xy + y^2)(x + y)(x^2 - xy + y^2)$

$=(x^3 - y^3)(x^2 + y^3) = \boldsymbol{x^6 - y^6}$

◂ Cubic Formula #2(p.20)

(10)

$(a + b)^2(a - b)^2(a^4 + a^2b^2 + b^4)^2$

$=\left\{(a + b)(a - b)(a^4 + a^2b^2 + b^4)\right\}^2$

$=\left\{(a^2 - b^2)(a^4 + a^2b^2 + b^4)\right\}^2$

$=(a^6 - b^6)^2 = \boldsymbol{a^{12} - 2a^6b^6 + b^{12}}$

◂ Formulas: Product of addition and subtraction of integral expressions(p.17)

◂ Cubic Formula #2(p.20)

1.2.5 Fundamental factorization

Instead of expanding integral expressions, dividing expression A into two expressions B and C of the product form is called **factorization**. B and C in this case, are called the **factors** of A.When factoring expressions, if there is a **common factor**, put it out side the parentheses as shown in the example problems.

【Example Questions: Factorizing with common factors】

Factorize the following expressions.

(1) $a^4 - 2a$ (2) $6a^2b + 4ab^2 - 2ab$

(3) $p(2x - y) + q(y - 2x)$ (4) $3a(x - y) + 6b(x - y) + 9c(y - x)$

【Answers】

(1) Since a is the common factor of a^4 and $-2a$,

$a^4 - 2a = \boldsymbol{a(a^3 - 2)}$

◂ Factor by common factor of a

	a^3	-2
a	a^4	$-2a$

(2) Since $2ab$ is the common factor of $6a^2b$, $4ab^2$ and $-2ab$

$6a^2b + 4ab^2 - 2ab = \boldsymbol{2ab(3a + 2b - 1)}$

◂ Factor with the common factor $2ab$

	$3a$	$2b$	-1
$2ab$	$6a^2b$	$4ab^2$	$-2ab$

(3) Since $y - 2x = -(2x - y)$, the whole $2x - y$ is the common factor

$p(2x - y) + q(y - 2x)$

$=p(2x - y) - q(2x - y)$

◂ Factor with the common factor $2x - y$

	p	$-q$
$2x - y$	$p(2x - y)$	$-q(2x - y)$

$=\mathbf{(2x - y)(p - q)}$

(4) Since $y - x = -(x - y)$, the whole $x - y$ is the common factor

$3a(x-y) + 6b(x-y) + 9c(y-x)$
$=3a(x-y) + 6b(x-y) - 9c(x-y)$
$=3(x-y)\cdot a + 3(x-y)\cdot 2b - 3(x-y)\cdot 3c$
$=\mathbf{3(x - y)(a + 2b - 3c)}$

◀ Factor with the common factor $3(x-y)$

	a	$2b$	$-3c$
$3(x-y)$	$3a(x-y)$	$6b(x-y)$	$-9c(x-y)$

Although there may not be a common factor to factorize, it is still possible to factorize by reversing the algebraic formulas mentioned in the previous lessons.

■Factorizing by reversing the Squared Formulas (p.17)

【Example Problems: Factorizing by reversing the squared formulas】

Factorize the following expressions

(1) $x^2 + 6x + 9$　(2) $a^2 - 8a + 16$
(3) $4x^2 + 12xy + 9y^2$　(4) $9a^2 - 24a + 16$

【Answers】

(1)
$x^2 + 6x + 9$
$=x^2 + 2\cdot 3x + 3^2$
$=\mathbf{(x + 3)^2}$

◀

	x	3
x	x^2	$3x$
3	$3x$	9

(2)
$a^2 - 8x + 16$
$=a^2 - 2\cdot 4x + 4^2$
$=\mathbf{(a - 4)^2}$

◀

	x	-4
x	x^2	$-4x$
-4	$-4x$	16

(3)
$4x^2 + 12xy + 9y^2$
$=(2x)^2 + 2\cdot 6xy + (3y)^2$
$=\mathbf{(2x + 3y)^2}$

◀

	$2x$	3
$2x$	$4x^2$	$6x$
3	$6x$	9

(4)
$9a^2 - 24a + 16$
$=(3a)^2 - 2\cdot 12a + 4^2$
$=\mathbf{(3a - 4)^2}$

◀

	$3a$	-4
$3a$	$9a^2$	$-12a$
-4	$-12a$	16

Reversed squared formulas

1° $a^2 + 2ab + b^2 = (a+b)^2$

2° $a^2 - 2ab + b^2 = (a-b)^2$

After factorizing an expression, expand it again to check your answer. It is a little time consuming, but be sure to check afterwards.

About **Double Radical Sign**:

When $a > 0$, $b > 0$, $\left(\sqrt{a} + \sqrt{b}\right)^2 = a + b + 2\sqrt{ab}$, and since $\sqrt{a} + \sqrt{b} > 0$

$$\sqrt{a+b+2\sqrt{ab}} = \sqrt{\left(\sqrt{a}+\sqrt{b}\right)^2} = \sqrt{a} + \sqrt{b} \quad \cdots\cdots ①$$

Also when $a > b > 0$, $\left(\sqrt{a} - \sqrt{b}\right)^2 = a + b - 2\sqrt{ab}$, and since $\sqrt{a} - \sqrt{b} > 0$

$$\sqrt{a+b-2\sqrt{ab}} = \sqrt{\left(\sqrt{a}-\sqrt{b}\right)^2} = \sqrt{a} - \sqrt{b} \quad \cdots\cdots ②$$

Using ① and ②, the following example can be easily solved. For example, $\sqrt{8+2\sqrt{15}}$ is solved by using ①.

$$\sqrt{8+2\sqrt{15}} = \sqrt{(5+3)+2\sqrt{5\cdot 3}} = \sqrt{\left(\sqrt{5}+\sqrt{3}\right)^2} = \sqrt{5}+\sqrt{3}$$

So the squaring is taken off.

【Example problems: excluding squares】

Take off the following squares.

(1) $\sqrt{7+4\sqrt{3}}$ (2) $\sqrt{9-2\sqrt{14}}$ (3) $\sqrt{3+\sqrt{5}}$

【Answers】

(1)

$$\sqrt{7+4\sqrt{3}} = \sqrt{7+2\sqrt{12}}$$

◂ When taking two roots out, make into this shape

$$= \sqrt{\left(\sqrt{4}+\sqrt{3}\right)^2}$$

◂ Squared

$$= \sqrt{4}+\sqrt{3} = \mathbf{2+\sqrt{3}}$$

◂ Double root taken out

(2)

$$\sqrt{9-2\sqrt{14}} = \sqrt{\left(\sqrt{7}-\sqrt{2}\right)^2}$$

$$= \mathbf{\sqrt{7}-\sqrt{2}}$$

◂ $(○-△)^2$ を作るときには，○ > △ を満たすように作るとよい

(3)

$$\sqrt{3+\sqrt{5}} = \sqrt{3+\sqrt{5}} \times \frac{\sqrt{2}}{\sqrt{2}}$$

◂ although there is no 2 under the root sign, you can multiply root 2 to both the numerator and the denominator in order to make the form 2 root.

$$= \frac{\sqrt{6+2\sqrt{5}}}{\sqrt{2}}$$

$$= \frac{\sqrt{\left(\sqrt{5}+\sqrt{1}\right)^2}}{\sqrt{2}}$$

$$= \frac{\sqrt{5}+1}{\sqrt{2}} = \mathbf{\frac{\sqrt{10}+\sqrt{2}}{2}}$$

◀ Keep the denominator rational.

■Factorization using the reversed Formulas: Product of addition and subtraction of integral expressionson p.17

【Example Questions: Factorization using the reversed Formulas: Product of addition and subtraction of i

Factorize the following expressions.

(1) $a^2 - 9$ (2) $4x^2 - 25y^2$

(3) $(a-b)^2 - c^2$ (4) $4x^2 - 9(a-b)^2$

【Answerss】

(1)

$a^2 - 9$

$=a^2 - 3^2$

$=\mathbf{(a+3)(a-3)}$

	a	-3
a	a^2	$-3a$
3	$3a$	-9

(2)

$4x^2 - 25y^2$

$=(2x)^2 - (5y)^2$

$=\mathbf{(2x+5y)(2x-5y)}$

	$2x$	$-5y$
$2x$	$4x^2$	$-10xy$
$5y$	$10xy$	$-25y^2$

(3)

$(a-b)^2 - c^2$

$=\{(a-b)+c\}\{(a-b)-c\}$

$=\mathbf{(a-b+c)(a-b-c)}$

	$a-b$	$-c$
$a-b$	$(a-b)^2$	$-(a-b)c$
c	$(a-b)c$	$-c^2$

(4)

$4x^2 - 9(a-b)^2$

$=(2x)^2 - \{3(a-b)\}^2$

$=\{2x+3(a-b)\}\{2x-3(a-b)\}$

$=\mathbf{(2x+3a-3b)(2x-3a+3b)}$

	$2x$	$-3(a-b)$
$2x$	$4x^2$	$-6x(a-b)$
$3(a-b)$	$6x(a-b)$	$-9(a-b)^2$

Reversed formula: product of addition and subtraction of integral expressions

3° $a^2 - b^2 = (a+b)(a-b)$

Notice quickly that you are able to factorize when the expression is in this shape: ○^{2} − △^{2}

■Reversed factorization of Product of Linear Equation(p.18)

Let's use $x^2 + 5x + 6$ for an easy example to start with and factorize step by step

STEP1

To consider the expanded linear expression $x^2 + 5x + 6$, draw the table shown at the right. Just like the table, fill in the diagonal with x^2, then the constant term.

	$\boldsymbol{x^2}$	
		6

STEP2

x^2 becomes $x \times x$ when expanded, 6 may be factorized by 1×6 or 2×3, so two tables are created. ($(-1) \times (-6)$ or $(-2) \times (-3)$ for factorizing 6 is unlikely in this case since the term $5x$ is positive).

	$\boldsymbol{x}$	**6**
$\boldsymbol{x}$	x^2	
1		6

	$\boldsymbol{x}$	**3**
$\boldsymbol{x}$	x^2	
2		6

STEP3

Fill in the other blanks to complete the table. By using the table that sums up to $5x$ when the bold terms are added, factorization is completed.

$x^2 + 5x + 6 = (x + 2)(x + 3)$

×	x	6
x	x^2	$\boldsymbol{6x}$
1	$\boldsymbol{x}$	6

○	x	3
x	x^2	$\boldsymbol{3x}$
2	$\boldsymbol{2x}$	6

【Example Questions: Reversed factorization of Linear Equations #1】

Factorize the following expressions.

(1) $x^2 + 10x + 21$ (2) $x^2 - 6x + 8$

(3) $a^2 + 3ab - 18b^2$ (4) $a^2 - 4a - 32$

【Answers】

◂ Tables below are some examples of tables when factorizing

(1)

$x^2 + 10x + 21$

$= x^2 + (3 + 7)x + 3 \cdot 7$

$= \boldsymbol{(x + 3)(x + 7)}$

◂

×	x	21
x	x^2	$21x$
1	x	21

○	x	7
x	x^2	$7x$
3	$3x$	21

(2)

$x^2 - 6x + 8$

$= x^2 + (-2 - 4)x + (-2)(-4)$

$= \boldsymbol{(x - 2)(x - 4)}$

◂

×	x	-8
x	x^2	$-8x$
-1	$-x$	8

○	x	-4
x	x^2	$-4x$
-2	$-2x$	8

(3)

$a^2 + 3ab - 18b^2$

$= a^2 + (-3 + 6)ab + (-3b)(6b)$

$= \mathbf{(a - 3b)(a + 6b)}$

◄

×	a	$-18b$
a	a^2	$-18ab$
$-b$	$-ab$	$-18b^2$

×	a	$9b$
a	a^2	$9ab$
$-2b$	$-2ab$	$-18b^2$

○	a	$6b$
a	a^2	$6ab$
$-3b$	$-3ab$	$-18b^2$

(4)

$a^2 - 4a - 32$

$= a^2 + (4 - 8)a + 4 \cdot (-8)$

$= \mathbf{(a + 4)(a - 8)}$

◄

○	a	-8
a	a^2	$-8a$
4	$4a$	-32

Let's try a harder example with $3x2 + 11x + 6$.

STEP1

Similar to the example before, fill in the table as shown at the right, with the highest degree term, then the constant term diagonally.

	$\mathbf{3x^2}$	
		6

STEP2

Since $3x^2$ factorizes to $x \times 3x$ and 6 is factorized either by 1×6 or 2×3, four tables are created as shown at the right. (It is possible that 6 can be factorized by $(-1) \times (-6)$ or $(-2) \times (-3)$ but since the coefficient of degree 1 of term x is 11, these factors are not possible).

	$3x$	6
x	$3x^2$	
1		6

	$3x$	3
x	$3x^2$	
2		6

	$3x$	2
x	$3x^2$	
3		6

	$3x$	1
x	$3x^2$	
6		6

STEP3

Fill in the rest of the tables and pick the one that has a sum of 11 for the coefficient of degree 1 of x and use that table to complete the factorization of the expression.

$3x^2 + 11x + 6 = (x + 3)(3x + 2)$

×	$3x$	6
x	$3x^2$	$6x$
1	$3x$	6

×	$3x$	3
x	$3x^2$	$3x$
2	$6x$	6

○	$3x$	2
x	$3x^2$	$2x$
3	$9x$	6

×	$3x$	1
x	$3x^2$	x
6	$18x$	6

When you get the hang of this table method, you will not have to make many tables as shown on the example above. You will later be able to multiply terms with one try.

【Example Questions: Reversed factorization of Linear Equations #2】

Factorize the following expressions

(1) $2x^2 + 3x + 1$　(2) $5a^2 + 7ab + 2b^2$

(3) $8x^2 - 10xy + 3y^2$　(4) $12a^2 + 7a - 12$

【Answers】

◀ Tables below are some examples of tables when factorizing

(1)

$2x^2 + 3x + 1$

$=2x^2 + (2 + 1)x + 1$

$=\mathbf{(x + 1)(2x + 1)}$

○	$2x$	1
x	x^2	x
1	$2x$	1

(2)

$5a^2 + 7ab + 2b^2$

$=5a^2 + (5 + 2)ab + 2b^2$

$=\mathbf{(a + b)(5a + 2b)}$

○	$5a$	$2b$
a	$5a^2$	$2ab$
b	$5ab$	$2b^2$

×	$5a$	b
a	$5a^2$	ab
$2b$	$10ab$	$2b^2$

(3)

$8x^2 - 10xy + 3y^2$

$=8x^2 + (-6 - 4)xy + 3y^2$

$=\mathbf{(2x - y)(4x - 3y)}$

×	$8x$	$-3y$
x	$8x^2$	$-3xy$
$-y$	$-8xy$	$3y^2$

×	$8x$	$-y$
x	$8x^2$	$-xy$
$-3y$	$-24xy$	$3y^2$

○	$4x$	$-3y$
$2x$	$8x^2$	$-6xy$
$-y$	$-4xy$	$3y^2$

×	$4x$	$-y$
$2x$	$8x^2$	$-2xy$
$-3y$	$-12xy$	$3y^2$

(4)

$12a^2 + 7a - 12$

$=12a^2 + (16 - 9)a - 12$

$=\mathbf{(3a + 4)(4a - 3)}$

×	$4a$	3
$3a$	$12a^2$	$9a$
-4	$-16a$	-12

○	$4a$	-3
$3a$	$12a^2$	$-9a$
4	$16a$	-12

Reversing Product of Linear Expressions

4° $x^2 + (b + d)x + bd = (x + b)(x + d)$

5° $acx^2 + (ad + bc)x + bd = (ax + b)(cx + d)$

■Reversed factorization using the formula for squaring three terms(p.19)

【Example Questions: Reversed factorization using the formula for squaring three terms】

Factorize the following expressions

(1) $a^2 + 4b^2 + c^2 + 4ab + 4bc + 2ca$ (2) $4x^2 + y^2 + 1 + 4xy + 2y + 4x$

(3) $4a^2 + b^2 + 1 + 4ab - 2b - 4a$ (4) $x^2 + 4y^2 + 9z^2 - 4xy + 12yz - 6zx$

【Answers】

(1)

$$a^2 + 4b^2 + c^2 + 4ab + 4bc + 2ca$$
$$= a^2 + (2b)^2 + c^2 + 2 \cdot a(2b) + 2 \cdot (2b)c + 2 \cdot ca$$
$$= \boldsymbol{(a + 2b + c)^2}$$

	a	$2b$	c
a	a^2	$2ab$	ac
$2b$	$2ab$	$4b^2$	$2bc$
c	ac	$2bc$	c^2

(2)

$$4x^2 + y^2 + 1 + 4xy + 2y + 4x$$
$$= (2x)^2 + y^2 + 1^2 + 2 \cdot (2x)y + 2 \cdot y \cdot 1 + 2 \cdot 1 \cdot x$$
$$= \boldsymbol{(2x + y + 1)^2}$$

	$2x$	y	1
$2x$	$4x^2$	$2xy$	$2x$
y	$2xy$	y^2	y
1	$2x$	y	1

(3)

$$4a^2 + b^2 + 1 + 4ab - 2b - 4a$$
$$= (2a)^2 + b^2 + (-1)^2$$
$$+ 2 \cdot (2a)b + 2 \cdot b(-1) + 2 \cdot (-1)a$$
$$= \boldsymbol{(2a + b - 1)^2}$$

	$2a$	b	-1
$2a$	$4a^2$	$2ab$	$-2a$
b	$2ab$	b^2	$-b$
-1	$-2a$	$-b$	1

(4)

$$x^2 + 4y^2 + 9z^2 - 4xy + 12yz - 6zx$$
$$= x^2 + (2y)^2 + (3z)^2$$
$$+ 2 \cdot x(-2y) + 2 \cdot (-2y)(-3z) + 2 \cdot (-3z)x$$
$$= \boldsymbol{(x - 2y - 3z)^2}$$

	x	$-2y$	$-3z$
x	x^2	$-2xy$	$-3xz$
$-2y$	$-2xy$	$4y^2$	$6yz$
$-3z$	$-3xz$	$6yz$	$9z^2$

Reversed usage of squaring three terms

6° $a^2 + b^2 + c^2 + 2ab + 2bc + 2ca = (a + b + c)^2$

■Reversed factorization using the cubic formula #1(p.19)

【Example Questions: Using reversed cubic formula #1 to factorize】

Factorize the following expressions.

(1) $x^3 + 9x^2y + 27xy^2 + 27y^3$

(2) $8a^3 + 12a^2b + 6ab^2 + b^3$

(3) $x^3 - 6x^2 + 12x - 8$

(4) $27x^3 - 54x^2y + 36xy^2 - 8y^3$

【Answers】

(1)

$$x^3 + 9x^2y + 27xy^2 + 27y^3$$
$$= x^3 + 3 \cdot x^2(3y) + 3 \cdot x(3y)^2 + (3y^3)$$
$$= \boldsymbol{(x + 3y)^3}$$

(2)

$$8a^3 + 12a^2b + 6ab^2 + b^3$$
$$= (2a)^3 + 3 \cdot (2a)^2b + 3 \cdot (2a)b^2 + b^3$$
$$= \boldsymbol{(2a + b)^3}$$

(3)

$$x^3 - 6x^2 + 12x - 8$$
$$= x^3 - 3 \cdot x^2 \cdot 2 + 3 \cdot x \cdot 2^2 - 2^3$$
$$= \boldsymbol{(x - 2)^3}$$

(4)

$$27x^3 - 54x^2y + 36xy^2 - 8y^3$$
$$= (3x)^3 - 3 \cdot (3x)^2(2y) + 3 \cdot (3x)(2y)^2 - (2y)^3$$
$$= \boldsymbol{(3x - 2y)^3}$$

Reversed cubic formula #1

7° $a^3 + 3a^2b + 3ab^2 + b^3 = (a + b)^3$

8° $a^3 - 3a^2b + 3ab^2 - b^3 = (a - b)^3$

With this kind of factorization, $\boldsymbol{a^3} + \bigcirc + \triangle + \boldsymbol{b^3}$, arrange the expression to $(a+b)^3$ cubed format first, then expand later to check your answer.

■Reversed factorization using the cubic formula #2(p.20)

【Reversed factorization using the cubic formula #2**】**

Factorize the following expressions

(1) $x^3 + 27$ (2) $8a^3 + 1$

(3) $8x^3 - 27y^3$ (4) $64a^3 - 125b^3$

【Answers】

(1)

$x^3 + 27$

$= x^3 + 3^3$

$= \mathbf{(x + 3)(x^2 - 3x + 9)}$

	x^2	$-3x$	9
◀ x	x^3	$-3x^2$	$9x$
3	$3x^2$	$-9x$	27

(2)

$8a^3 + 1$

$= (2a)^3 + 1^3$

$= \mathbf{(2a + 1)(4a^2 - 2a + 1)}$

	$4a^2$	$-2a$	1
◀ $2a$	$8a^3$	$-4a^2$	$2a$
1	$4a^2$	$-2a$	1

(3)

$8x^3 - 27y^3$

$= (2x)^3 - (3y)^3$

$= \mathbf{(2x - 3y)(4x^2 + 6xy + 9y^2)}$

	$4x^2$	$6xy$	$9y^2$
◀ $2x$	$8x^3$	$12x^2y$	$18xy^2$
$-3y$	$-12x^2y$	$-18xy^2$	$-27y^3$

(4)

$64a^3 - 125b^3$

$= (4a)^3 - (5b)^3$

$= \mathbf{(4a - 5b)(16a^2 + 20ab + 25b^2)}$

	$16a^2$	$20ab$	$25b^2$
◀ $4a$	$64a^3$	$80a^2b$	$100ab^2$
$-5b$	$-80a^2b$	$-100ab^2$	$-125b^3$

Reversed cubic formula #2

9° $a^3 + b^3 = (a + b)(a^2 - ab + b^2)$

10° $a^3 - b^3 = (a - b)(a^2 + ab + b^2)$

It is easy to forget this $\bigcirc^3 \pm \triangle^3$ type of factorization so be familiarized with it.

1.2.6 Complex Factorization

■Factorization of cubed expression with three variables

Mentioned before in cubic formula #1 (p.19), when $(a+b)3$ is expanded:

$$(a+b)^3 = a^3 + 3a^2b + 3ab^2 + b^3$$

So

$$a^3 + b^3 = (a+b)^3 - 3a^2b - 3ab^2$$

$$\Leftrightarrow\ a^3 + b^3 = (a+b)^3 - 3ab(a+b) \quad \cdots\cdots ①$$

is formulated. Accordingly, $a3 + b3 + c3 - 3abc$ can be factorized in the following:

【Memorize: Factorizing cubed expression with three variables】

Factorize $a^3 + b^3 + c^3 - 3abc$.

$$a^3 + b^3 + c^3 - 3abc$$

$$= \left\{a^3 + b^3\right\} + c^3 - 3abc$$ ◂ Notice $a^3 + b^3$

$$= \left\{(a+b)^3 - 3ab(a+b)\right\} + c^3 - 3abc$$ ◂ used ①

$$= (a+b)^3 - 3ab(a+b) + c^3 - 3abc$$

$$= \underbrace{(a+b)^3 + c^3}_{\bigcirc} \underbrace{-3ab(a+b) - 3abc}_{\square}$$ ◂ 今度は $(a+b)^3 + c^3$ に着目する

$$= \underbrace{\{(a+b)+c\}\left\{(a+b)^2 - (a+b)c + c^2\right\}}_{\bigcirc} \underbrace{-3ab\{(a+b)+c\}}_{\square}$$ ◂ the ○ part is factorized by making $a+b = X$, $c = y$ and using $X^3 + Y^3 = (X+Y)(X^2 - XY + Y^2)$ の因数分解 (p.33) を使った

$$= (a+b+c)\left\{a^2 + 2ab + b^2 - ac - bc + c^2\right\} - 3ab(a+b+c)$$

$$= (a+b+c)(a^2 + 2ab + b^2 - ac - bc + c^2 - 3ab)$$ ◂ Factorized by common factor $(a+b+c)$

$$= (a+b+c)(a^2 + b^2 + c^2 - ab - bc - ca)$$

Factorization of cubic expression with three variables

11° $a^3 + b^3 + c^3 - 3abc = (a+b+c)(a^2 + b^2 + c^2 - ab - bc - ca)$

Since this factorization is not easy to memorize, review the steps shown above to practice.

【Example Questions: Factorizing cubic expressions with three variables】

Factorize the following expressions.

(1) $27a^3 + 8b^3 + c^3 - 18abc$ (2) $x^3 + y^2 - 1 + 3xy$

【Answers】

(1)

$$(3a)^2 + (2b)^3 + c^2 - 3(3a)(2b)c$$
$$= \mathbf{(3a + 2b + c)(9a^2 + 4b^2 + c^2 - 6ab - 2bc - 3ca)}$$

(2)

$$x^3 + y^3 + (-1)^3 - 3xy(-1)$$
$$= \mathbf{(x + y - 1)(x^2 + y^2 + 1 - xy + y + x)}$$

■Factorization of compound quadratic expression

Although there are integral expressions that have x^2 as a single unit, we will be considering factorization of expressions that are called compound quadratic expressions.

Definition of Compound Quadratic Expression

When real numbers a, b, c are constant,

$$ax^4 + bx^2 + c$$

This form of expression is called **compound quadratic expression**, only when $a \doteqdot 0$.

For instance, consider the following two compound quadratic expressions.

i) $x^4 - 13x^2 + 36$ ii) $x^4 + 2x^2 + 9$

i) Factorize $x^4 - 13x^2 + 36$

First, when $x2 = X$, this expression can be simplified to $X^2 - 13X + 36 = (X - 4)(X - 9)$ and use this to factorize:

$$x^4 - 13x^2 + 36$$
$$= (x^2 - 4)(x^2 - 9)$$
$$= (x + 2)(x - 2)(x + 3)(x - 3)$$

ii) Factorize $x^4 + 2x^2 + 9$

In this case, when $x2 = X$, $X^2 + 2X + 9$ cannot be factorized.

So focus on x^4 and 9 to factorize:

$$x^4 + 2x^2 + 9$$
$$= x^4 + 6x^2 + 9 - 4x^2 \quad \leftarrow \text{Add } 4x^2 \text{ to } 2x^2 \text{ in order to make a square}$$
$$= \underbrace{(x^2 + 3)^2}_{\text{make into a square}} - (2x)^2 \quad \leftarrow \text{the form of} \bigcirc^2 - \triangle^2$$

$= \{(x^2+3)+2x\}\{(x^2+3)-2x\}$
$= (x^2+2x+3)(x^2-2x+3)$

Factorization of Compound Quadratic expression

Factorization of compound quadratic expression ax^4+bx^2+c by,

i) substituting X for x^2
ii) Focus on ax^4 and c to add another x^2 term

Can be concluded.

When i) does not work, remember ii) in order to solve the expressions. For detail, refer to p.**??**

【Example Questions: Factorizing Compound Quadratic Expressions】

Factorize the following expressions.

(1) x^4-7x^2-8 (2) x^4+x^2+1

【Answers】

(1) Substitute X for x^2,

(Given expression) $= X^2-7X-8$
$= (X+1)(X-8)$
$= (x^2+2x+1)(x^2+2x-8)$
$= \mathbf{(x+1)^2(x+4)(x-2)}$

(2) Focus on x^4 and 1,

(Given expression) $= x^4+2x^2+1-x^2$
$= (x^2+1)^2-x^2$
$= (x^2+1+x)(x^2+1-x)$
$= \mathbf{(x^2+x+1)(x^2-x+1)}$

■Factorization of quadratic expression with two variables.

Consider factorizing $x^2 + 4xy + 3y^2 + x + 5y - 2$.

【**Method** #1**: Draw a** 3 − **by** − 3 **table**】

STEP1

When factorizing

$\boldsymbol{x^2} + 4xy \boldsymbol{+3y^2} + x + 5y \boldsymbol{-2}$

transfer the term x^2 (x^2), term y^2 ($3y^2$), and the constant term (−2) to the table as shown at the right.

	$\boldsymbol{x^2}$		
		$\boldsymbol{3y^2}$	
			−2

STEP2

Start filling in the outer left corner, like the table at the right.

First start with x^2.

	$\boldsymbol{x}$		
$\boldsymbol{x}$	x^2		
		$3y^2$	
			−2

STEP3

As shown at the right, $3y$ and y is inserted since $3y^2$ can be factorized. When xy and $3xy$ forms, the sum of these two terms should equal the $4xy$ in the expression. ($3y^2 = (-3y) times (-y)$ does not have to be considered since the sum of the terms will be $-4xy$).

x^2 $\boldsymbol{+4xy}$ $+3y^2 + x + 5y - 2$

	x	y	
x	x^2	$\boldsymbol{xy}$	
$3y$	$\boldsymbol{3xy}$	$3y^2$	
			−2

STEP4

Lastly, factorize the constant term (−2).

Lastly, factorize the constant term (−2). The two tables on the right with x and $-2x$ does not equal $+x$ in the expression $x2 + 4xy + 3y2 + x + 5y - 2$ so these two are omitted. Then fill in the rest of the table to find the one that equals $+5y$, which leads to the answer; $(x + 3y - 1)(x + y + 2)$.

×	x	y	1
x	x^2	xy	$\boldsymbol{x}$
$3y$	$3xy$	$3y^2$	
−2	$\boldsymbol{-2x}$		−2

×	x	y	−2
x	x^2	xy	$\boldsymbol{-2x}$
$3y$	$3xy$	$3y^2$	
1	$\boldsymbol{x}$		−2

×	x	y	−1
x	x^2	xy	$-x$
$3y$	$3xy$	$3y^2$	$\boldsymbol{-3y}$
2	$2x$	$\boldsymbol{2y}$	−2

○	x	y	2
x	x^2	xy	$2x$
$3y$	$3xy$	$3y^2$	$\boldsymbol{6y}$
−1	$-x$	$\boldsymbol{-y}$	−2

【**Method** #2**: First Arranging the expression in the descending order of power, then**

making the table】

First, arrange the expression $x^2 + 4xy + 3y^2 + x + 5y - 2$ as an expression of x, in the descending order of powers.

$$x^2 + (4y + 1)x + 3y^2 + 5y - 2$$

Then factorize the terms that do not contain x.

$$x^2 + (4y + 1)x + (3y - 1)(y + 2)$$

Similar to reversing the product of linear expression (p.28), create tables, like the ones below and fill in by order.

	x	$y+2$
x	x^2	
$3y-1$		$(3y-1)(y+2)$

→

○	x	$y+2$
x	x^2	$(y+2)x$
$3y-1$	$(3y-1)x$	$(3y-1)(y+2)$

so

$$(x + y + 2)(x + 3y - 1)$$

Although this method may seem easier to solve, it is actually more painstaking when x^2 has a coefficient. On the other hand, method #1 has only one step; thus, use method #1 first to improve your factorization skills.

【Example Questions: Factorization of quadratic expression with two variables】

Factorize the following expressions.

(1) $2x^2 + 5xy + 3y^2 + 2x + 4y - 4$　　(2) $6x^2 - 5xy - 6y^2 + 4x + 7y - 2$

【Answers】

(1) 【method 1]】

On the 3 – by – 3 table;

	x		
$2x$	$2x^2$		
		$3y^2$	
			-4

→

	x	y	
$2x$	$2x^2$	$2xy$	
$3y$	$3xy$	$3y^2$	
			-4

→

	x	y	2
$2x$	$2x^2$	$2xy$	$4x$
$3y$	$3xy$	$3y^2$	$6y$
-2	$-2x$	$-2y$	-4

Is created so,

$$2x^2 + 5xy + 3y^2 + 2x + 4y - 4$$
$$= \mathbf{(2x + 3y - 2)(x + y + 2)}$$

【method 2】

Arrange the expression in terms of x in the descending order of power.

$$2x^2 + 5xy + 3y^2 + 2x + 4y - 4$$

$= 2x^2 + (5y + 2)x + 3y^2 + 4y - 4$

$= 2x^2 + (5y + 2)x + (3y - 2)(y + 2)$

So the table

	x	
$2x$	$2x^2$	
		$(3y - 2)(y + 2)$

→

	x	$y + 2$
$2x$	$2x^2$	$2x(y + 2)$
$3y - 2$	$(3y - 2)x$	$(3y - 2)(y + 2)$

Is created.

$2x^2 + 5xy + 3y^2 + 2x + 4y - 4$

$= \mathbf{(2x + 3y - 2)(x + y + 2)}$

(2) 【method 1】

On the 3 – by – 3 table;

	$3x$		
$2x$	$6x^2$		
		$-6y^2$	
			-2

→

	$3x$	$2y$	
$2x$	$6x^2$	$4xy$	
$-3y$	$-9xy$	$-6y^2$	
			-2

→

	$3x$	$2y$	-1
$2x$	$6x^2$	$4xy$	$-2x$
$-3y$	$-9xy$	$-6y^2$	$3y$
2	$6x$	$4y$	-2

Is created so

$6x^2 - 5xy - 6y^2 + 4x + 7y - 2$

$= \mathbf{(2x - 3y + 2)(3x + 2y - 1)}$

【method 2】

Arrange the expression in terms of x in the descending order of power.

$6x^2 - 5xy - 6y^2 + 4x + 7y - 2$

$= 6x^2 - (5y - 4)x - (6y^2 - 7y + 2)$

$= 6x^2 - (5y - 4)x - (3y - 2)(2y - 1)$

So on the table,

	$3x$	
$2x$	$6x^2$	
		$(3y - 2)(2y - 1)$

→

	$3x$	$2y - 1$
$2x$	$6x^2$	
$-(3y - 2)$		$-(3y - 2)(2y - 1)$

Is created.

$6x^2 - 5xy - 6y^2 + 4x + 7y - 2$

$= \mathbf{(2x - 3y + 2)(3x + 2y - 1)}$

【Example Questions: Practicing Factorization】

Factorize the following expressions.

(1) $x^8 - 1$
(2) $a^6 - b^6$
(3) $5(x+y)^2 - 8(x+y) - 4$
(4) $(a+b)^2 + 10c(a+b) + 25c^2$
(5) $xy - x - y + 1$
(6) $a^2 + b^2 + ac - bc - 2ab$
(7) $x^2(y-z) + y^2(z-x) + z^2(x-y)$
(8) $ab(a-b) + bc(b-c) + ca(c-a)$
(9) $x^4 + x^2 + 1$
(10) $a^4 + 64$
(11) $x^2 - xy - 12y^2 + 5x + y + 6$
(12) $2x^2 - y^2 - xy + 3x + 3y - 2$

【Answers】

(1)

$(x^8 - 1)$

$= (x^4+1)(x^4-1)$ ◀ Reversed squared formulas(p.25)

$= (x^4+1)(x^2+1)(x^2-1)$ ◀ Reversed squared formulas(p.25)

$= \mathbf{(x^4+1)(x^2+1)(x+1)(x-1)}$ ◀ Reversed squared formulas(p.25)

(2)

$a^6 - b^6$

$= (a^3+b^3)(a^3-b^3)$ ◀ Reversed squared formulas(p.25)

$= (a+b)(a^2-ab+b^2)(a-b)(a^2+ab+b^2)$ ◀ Reversed cubic formula 2(p.33)

$= \mathbf{(a+b)(a-b)(a^2-ab+b^2)(a^2+ab+b^2)}$

(3) Substitute X for $x+y$

$5X^2 - 8X - 4$

$= (5X+2)(X-2)$ ◀ Reversed factorization of Product of Linear Equation(p.28)

$= \mathbf{(5x+5y+2)(x+y-2)}$

(4) Substitute X for $a+b$

$X^2 + 10cX + 25c^2$

$= (X+5c)^2$ ◀ Reversed squared formulas(p.25)

$= \mathbf{(a+b+5c)^2}$

(5) Arrange it in terms of x in the descending order of power.

$(y-1)x - (y-1)$

$= \mathbf{(x-1)(y-1)}$

(6) Arrange it in terms of c in the descending order

of power.

$$(a-b)c+a^2+b^2-2ab$$
$$=(a-b)c+(a-b)^2$$
$$=(a-b)c+(a-b)(a-b)$$
$$=\boldsymbol{(a-b)(a-b+c)}$$

◂ Organizing by the terms with the lowest degree can be helpful

(7) Arrange it in terms of x in the descending order of power.

◂ Notice the expression as an expression of x

$$(y-z)x^2-(y^2-z^2)x+y^2z-yz^2$$
$$=(y-z)x^2-(y+z)(y-z)+yz(y-z)$$
$$=(y-z)\left\{x^2-(y+z)x+yz\right\}$$
$$=\boldsymbol{(y-z)(x-y)(x-z)}$$

(8) Arrange it in terms of a in the descending order of power.

◂ Notice the expression as an expression of a

$$(b-c)a^2-(b^2-c^2)a+b^2c-bc^2$$
$$=(b-c)a^2-(b+c)(b-c)a+bc(b-c)$$
$$=(b-c)\left\{a^2-(b+c)a+bc\right\}$$
$$=\boldsymbol{(b-c)(a-b)(a-c)}$$

(9) Focus on x^4 and 1

◂ Factorization of compound quadratic expression(p.35)

$$x^4+1+x^2$$
$$=(x^2+1)^2-2x^2+x^2$$
$$=(x^2+1)^2-x^2$$
$$=(x^2+1+x)(x^2+1-x)$$
$$=\boldsymbol{(x^2+x+1)(x^2-x+1)}$$

(10) Focus on a^4 and 64

◂ Factorization of compound quadratic expression(p.35)

$$(a^2+8)^2-16a^2$$
$$=(a^2+8+4a)(a^2+8-4a)$$
$$=\boldsymbol{(a^2+4a+8)(a^2-4a+8)}$$

(11) Draw a 3 – by – 3 table

◂ Factorization of cubed expression with three variables(p.34)

	x	$-3y$	2
x	x^2	$3xy$	$2x$
$-4y$	$-4xy$	$-12y^2$	$-8y$
3	$3x$	$9y$	6

which can be factorized into

$\boldsymbol{(x-4y+3)(x+3y+2)}$

(12) Draw a 3 – by – 3 table

	x	$-y$	2
$2x$	$2x^2$	$-2xy$	$4x$
y	xy	$-y^2$	$2y$
-1	$-x$	y	-2

which can be factorized into

$\boldsymbol{(x-4y+3)(x+3y+2)}$

◀ Factorization of cubed expression with three variables(p.34)